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Vertical oscillations in a viscous and thermally conducting isothermal atmosphere

By PATRICK LYONS
Grumman Aerospace Corporation, Bethpage, New York 11714

AND MICHAEL YANOWITCH
Department of Mathematics, Adelphi University, Garden City, Long Island, New York 11530

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The presence of dissipation in an isothermal atmosphere may cause upward-propagating small amplitude waves to be reflected downward. For an atmosphere with small dynamic viscosity μ this was demonstrated in Yanowitch (1967*b*); this will be referred to as case II. Here two problems will be investigated: (i) a thermally conducting atmosphere with small conductivity k (case III) and (ii) a viscous and thermally conducting atmosphere with small k and μ, and a small ratio μ/k, i.e. small Prandtl number (case IV). It will be shown that the validity of the model in case III is questionable. The solution for case IV is determined from the conditions that the average rate of energy dissipation and of entropy increase in a column of fluid be finite, but a radiation condition is required in case III. The solution for case III does not approximate the one for case IV uniformly, and the reflexion coefficient for case IV does not tend to the one for case III as the Prandtl number $Pr \to 0$, but varies periodically with $\log Pr$. Numerical results show that when the Prandtl number is not small the reflexion coefficient can be approximated by the asymptotic value obtained from case II.

1. Introduction

It is well known that upward-travelling atmospheric waves of small amplitude can be reflected downward if the Brunt–Väisälä frequency varies with the altitude z. But, even when the Brunt–Väisälä frequency is constant, additional reflexion is possible owing to dissipative effects which grow exponentially with the altitude owing to the decrease in the density. Although this type of reflexion is usually of lesser importance, it may be significant when the vertical wavelength is large.

Reflexion produced in a density-stratified fluid by an exponentially increasing kinematic viscosity $\nu = \mu/\rho$ was examined by Yanowitch (1967*a, b*). As is to be expected, the effect of viscosity is negligible in the region where ν is small (except possibly in a boundary layer), and the waves approximate those in an inviscid fluid. In the region where ν is large, on the other hand, the effect of viscosity is dominant and the motion dies out as $z \to \infty$. The reflexion is produced in the layer which connects these two regions. As $\mu \to 0$ the magnitude of the reflexion coefficient (the ratio of the amplitude of the incident wave to that of the reflected wave) approaches a limiting value: $|K_R| \to \exp(-2\pi^2 H/\lambda)$, where λ is the vertical wavelength and H is the density scale height. The reflexion is therefore greatest as $\lambda \to \infty$, and is negligible when the wavelength is small. Similar results were obtained by Lindzen (1968) for the problem of atmospheric tidal oscillations in which the dissipation is provided by Newtonian cooling.

These results cannot be obtained directly from the small amplitude inviscid problem ($\mu = 0$) since it contains no reflecting mechanism, and the radiation condition is appropriate in this case (see Lamb 1932, pp. 541–543). Moreover, the linear inviscid problem is not a reasonable one anyway, since the solutions obtained grow exponentially as $z \to \infty$, thus violating the small amplitude assumption. The approximation obtained by setting $\mu = 0$ cannot be valid uniformly for all z.

In this paper we shall examine separately and together the reflecting effects of viscosity and thermal conduction in an isothermal atmosphere. Nonlinear effects will be ignored although they may be important owing to the exponential growth with z of the oscillation amplitude. A numerical study of the nonlinear equations is given in Yanowitch (1969). A nonlinear treatment of waves in a weakly stratified atmosphere, i.e. one in which the scale height is large compared with the vertical wavelength, was given by Drazin (1969) and by Grimshaw (1972).

For the sake of brevity, we shall refer to Lamb's problem, where both the viscosity μ and the thermal conductivity k are zero, as case I and to the problem for a viscous fluid ($\mu > 0$, $k = 0$) previously referred to as case II. First we shall study the vertical wave motions in an inviscid isothermal atmosphere with small thermal conductivity k; this will be labelled case III. This problem is a natural one to investigate as a simple model for an atmosphere with small Prandtl number Pr (which is proportional to μ/k), such as the solar corona (see, for example, Parker 1963, pp. 37–38). Despite the presence of dissipation in both cases the solutions for case III differ markedly from those of case II. As is to be expected, the wave motion in the region where the thermal diffusivity is small approximates the wave motion of case I. In contrast to case II, the region of large thermal diffusivity is also capable of supporting wave motions, albeit of a different wavelength, and this necessitates the imposition of a radiation condition. The temperature oscillations die out as $z \to \infty$, but the oscillations in the vertical velocity increase exponentially just as they do in case I. Thus, wave energy in the region of small thermal diffusivity is partly reflected and partly transmitted with a changed wavelength into the region of large thermal diffusivity. Since some of the energy is carried off to infinity, it is not surprising that the limiting value of the reflexion coefficient $|K_R|$ is now smaller than for case II.

The exponential increase of the velocity in case III raises questions concerning the validity of the results, and to resolve these we shall examine case IV: the problem of a viscous and thermally conducting isothermal atmosphere ($\mu > 0$, $k > 0$) for small Prandtl number. This is a singular perturbation problem, since the solutions for case III do not approximate those of case IV uniformly, and the solution is found by matching two asymptotic approximations which are valid in overlapping regions. It is shown that in this problem there are two reflecting layers, the lower one due to the effect of thermal conduction and the upper one due to viscosity. The upper reflecting layer tends to $z = \infty$ as $Pr \to 0$ (k = constant, $\mu \to 0$), and below this layer the solution behaves like some solution of the problem in case III, while above it the energy dies out owing to viscous dissipation as it does in case II. Since the solution oscillates between the two layers, a shift in the upper layer of half a wavelength ought to leave the reflexion coefficient unchanged, and it is found indeed that $|K_R|$ varies periodically with log Pr. Therefore, the results of case III are not recovered in the limit in which $k \to 0$ and Pr is small. However, the value of $|K_R|$ obtained in case III is found to be approximately equal to the average value of $|K_R|$ for case IV.

The results of numerical computations are described in § 5. It turns out that the variation of $|K_R|$ with Pr can be approximated by the asymptotic results obtained for case II and for case IV. The formula for case IV appears to be quite accurate for Pr less than about 0·2, while above that $|K_R|$ is nearly constant and equal to $\exp(-2\pi^2 H/\lambda)$.

2. Statement of the problem

Suppose that a perfect gas which is viscous and thermally conducting occupies the upper half-space $z > 0$. We shall study small vertical oscillations about equilibrium, i.e. oscillations which depend only on the vertical co-ordinate z and on the time t.

Let p_0, ρ_0 and T_0 represent the equilibrium pressure, density and temperature. They are connected by the gas law $p_0 = \rho_0 R T_0$ and the hydrostatic relation $p_0' + g\rho_0 = 0$ (R = gas constant and g = acceleration due to gravity). We shall consider an isothermal atmosphere, i.e. T_0 = constant. Then, as is well known,

$$p_0(z) = p_0(0)\exp(-z/H), \quad \rho_0(z) = \rho_0(0)\exp(-z/H),$$

where $H = RT_0/g$ is the density scale height.

Let p, ρ, T and w denote the perturbations in the pressure, density, temperature and vertical velocity. The linearized equations of motion (conservation of momentum and mass, the heat flow equation and the gas law) can be written in the form

$$\rho_0 w_t + p_z + g\rho = \tfrac{4}{3}\mu w_{zz}, \tag{2.1a}$$

$$\rho_t + (\rho_0 w)_z = 0, \tag{2.1b}$$

$$\rho_0(c_v T_t + gHw_z) = kT_{zz}, \tag{2.1c}$$

$$p = R(T_0\rho + \rho_0 T). \tag{2.1d}$$

Here μ is the dynamic viscosity coefficient, which is assumed to be constant, k is the thermal conductivity and c_v is the specific heat at constant volume. It is convenient to rewrite the problem in dimensionless form. Let

$$z^* = z/H, \quad t^* = \sigma_1 t, \quad w^* = w/c, \quad T^* = T/(2\gamma T_0),$$

where $c = (\gamma g H)^{\frac{1}{2}}$ is the speed of sound, with γ the ratio of specific heats, and $\sigma_1 = c/2H$. In addition we introduce two dimensionless parameters:

$$\epsilon = 2k/c_v\rho_0(0)\,cH \tag{2.2}$$

and the Prandtl number

$$Pr = c_p\mu/k, \tag{2.3}$$

where c_p is the specific heat at constant pressure. Eliminating the pressure and the density in (2.1) results in a pair of equations for T and w:

$$\epsilon e^z T_{zz} - T_t - [(\gamma-1)/\gamma]\, w_z = 0, \tag{2.4a}$$

$$3\gamma w_{tt} - 12(w_{zz} - w_z) - 4Pr\,\epsilon e^z w_{zzt} + 12\gamma(T_z - T)_t = 0. \tag{2.4b}$$

The asterisks have been dropped since only dimensionless variables will be considered from now on. Letting $w(z,t) = \tilde{W}(z)\,e^{-i\sigma t}$ and $T(z,t) = \tilde{T}(z)\,e^{-i\sigma t}$, substituting in (2.4) and dropping the tilde yields two equations for the complex amplitudes $W(z)$ and $T(z)$:

$$(\epsilon e^z D^2 + i\sigma)\,T - [(\gamma-1)/\gamma]\,DW = 0, \tag{2.5a}$$

$$(D^2 - D + \tfrac{1}{4}\gamma\sigma^2)\,W - \tfrac{1}{3}i\sigma Pr\,\epsilon e^z D^2 W + i\sigma\gamma(D-1)\,T = 0, \tag{2.5b}$$

where D denotes the derivative with respect to z. If, furthermore, W is eliminated from (2.5), one obtains a single fourth-order equation for T:

$$\{3i\sigma\gamma(D^2 - D + \tfrac{1}{4}\sigma^2) + 3\epsilon e^z D^2(D^2 + D + \tfrac{1}{4}\gamma\sigma^2) + \sigma^2 Pr\,\epsilon e^z\,(D^2 + D) \\ - (i\sigma Pr/\gamma)\,(\epsilon e^z)^2 D^2 (D+1)\,(D+2)\}\,T = 0. \tag{2.6}$$

We shall make use of both (2.5) and (2.6).

Boundary conditions. The oscillations can be assumed to be excited by some mechanism at $z = 0$ or below. The exact nature of the excitation is not important since our object is to study the reflexion which, for small ϵ, takes place at large heights (in the region near $z = -\log\epsilon$). For numerical computations (see § 5) we shall prescribe

$$W(0) = 1, \quad T(0) = 0. \tag{2.7}$$

Otherwise it is more convenient to adopt the condition that *in a fixed interval* $0 \leqslant z \leqslant z_1$, *the solution must approach some solution of the inviscid and non-conducting problem* ($\mu = k = 0$) *as* $\epsilon \to 0$. We shall refer to this as the *lower boundary condition*. Except for a normalizing constant, the solution obtained with the lower boundary condition will differ from the one obtained with (2.7) only in a narrow thermal boundary layer near $z = 0$, which has no effect on the limiting value of the reflexion coefficient.

Two further conditions which refer to the behaviour of solutions for large z are required. The first, which we shall call the *dissipation condition*, requires that a finite amount of energy be dissipated in an infinite column of fluid of unit cross-section per period of oscillation (see Yanowitch 1967*a*). Since the dissipation function depends on the squares of the velocity gradients, the dissipation condition is equivalent to

$$\int_0^\infty |W_z|^2\,dz < \infty \quad \text{if} \quad \mu > 0. \tag{2.8}$$

The dissipation condition is, of course, inapplicable if $\mu = 0$. The second condition is determined by the equation for the rate of change of entropy (see Landau & Lifshitz 1959, p. 186), from which it follows that

$$\int_0^\infty |T_z|^2 dz < \infty \quad \text{if} \quad k > 0. \tag{2.9}$$

The entropy condition is inapplicable if $k = 0$.

There are four possible cases.

Case I. $\mu = k = 0$. This is the classical problem considered in Lamb (1932, pp. 541–543). The equation for W is

$$(D^2 - D + \tfrac{1}{4}\sigma^2)\, W = 0,$$

a differential equation of second order with constant coefficients. For $\sigma > 1$ (i.e. $\sigma > \sigma_1$ in dimensional units) the solutions behave like $\exp\,(\frac{1}{2} \pm i\beta)\, z$, with $2\beta = (\sigma^2 - 1)^{\frac{1}{2}}$. Lamb's solution, satisfying $W(0) = 1$, is $W = \exp\,(\frac{1}{2} + i\beta)\, z$, an upward-propagating wave satisfying the radiation condition. As we have noted, this solution cannot be considered satisfactory since the growth of W as $z \to \infty$ violates the assumptions underlying the linearization.

Case II. $k = 0$, $\mu > 0$. This problem was considered in Yanowitch (1967*b*). The equation for W, which can be obtained from (2.5) by setting $\epsilon = 0$ and eliminating T, is

$$4(1 - i\delta e^z)\, D^2 W - 4DW + \sigma^2 W = 0,$$

where $\delta = 2\sigma\mu/3cH\rho_0(0)$. The equation for T is superfluous since $k = 0$. The problem has a unique solution satisfying the dissipation condition and the condition $W(0) = 1$. As $\delta \to 0$, the solution approaches a limit in a co-ordinate system which shifts so as to keep δe^z constant. This limit coincides with a solution of the inviscid problem in a region where $\delta e^z \ll 1$, but not with Lamb's solution, because of the appearance of a reflected wave. The magnitude of the reflexion coefficient is $|K_R| = e^{-\pi\beta}$. Thus, the radiation condition becomes more accurate as $\beta \to \infty$, i.e. as the vertical wavelength decreases. Furthermore, the solution is uniformly bounded for all z.

Case III. $\mu = 0$, $k > 0$, i.e. $Pr = 0$. This is the case of an inviscid but thermally conducting fluid, which might be expected to provide a natural approximation for the problem with small Prandtl number. A fourth-order differential equation is obtained by setting $Pr = 0$ in (2.6). Despite the presence of thermal conduction, the lower boundary condition and (2.9) do not suffice to determine a solution uniquely, and a radiation conduction must be imposed. Thus, the model suffers from the same drawback as the model in case I. Since part of the energy is reflected and part transmitted to infinity, the limiting value (as $\epsilon \to 0$) of the magnitude $|K_R|$ of the reflexion coefficient is smaller than for case II (see §3). The questionable validity of the results leads to the examination of the problem with small Prandtl number.

Case IV. $\mu > 0$, $k > 0$. The problem is to find a solution to the system of differential equations (2.5) satisfying (2.8), (2.9) and the lower boundary condition. Under these conditions the solution is unique and uniformly bounded for all

$z \geqslant 0$. An approximation for small Pr is obtained in §4 by a singular perturbation procedure, which shows that the solution for case III is not uniformly valid. The limiting value of $|K_R|$ (as $\epsilon \to 0$) is now found to depend on Pr. For $Pr < 0{\cdot}2$ the asymptotic results are found to agree well with those obtained by numerical integration (see §5).

3. Oscillations of an inviscid thermally conducting atmosphere

First we shall consider the case of an inviscid atmosphere with small thermal conductivity, and shall determine the asymptotic behaviour of the solution and the reflexion coefficient as $\epsilon \to 0$. The equation for T can be obtained by setting the Prandtl number $Pr = 0$ in (2.6):

$$\{i\sigma\gamma(D^2 - D + \tfrac{1}{4}\sigma^2) + \epsilon e^z D^2(D^2 + D + \tfrac{1}{4}\gamma\sigma^2)\} T = 0. \tag{3.1}$$

To solve this equation, it is convenient to introduce a new independent variable ξ defined by

$$\xi = e^{-z}/i\epsilon = \epsilon^{-1}\exp(-z + \tfrac{3}{2}i\pi), \tag{3.2}$$

which transforms (3.1) into

$$\{\gamma\sigma\xi(\theta^2 + \theta + \tfrac{1}{4}\sigma^2) - \theta^2(\theta^2 - \theta + \tfrac{1}{4}\gamma\sigma^2)\} T = 0; \tag{3.3}$$

here $\theta = \xi d/d\xi$. The point $\xi = 0$ corresponds to $z = \infty$, the point $\xi_0 = \epsilon^{-1}\exp(\tfrac{3}{2}i\pi)$ to $z = 0$ and the segment joining these points in the complex-ξ plane to $z \geqslant 0$. As $\epsilon \to 0$, the point ξ_0 tends to ∞. Consequently, we shall examine the asymptotic behaviour of solutions of (3.3) for large $|\xi|$.

The point $\xi = \infty$ is an irregular singular point of (3.3) of rank one. There exist four independent solutions, whose asymptotic behaviour as $\xi \to \infty$ along the ray with $\arg\xi = \tfrac{3}{2}\pi i$ is given by

$$\left.\begin{aligned} T_1^\infty(\xi) &\sim \xi^{\alpha_1}[1 + h_{11}\xi^{-1} + \ldots], \\ T_2^\infty(\xi) &\sim \xi^{\alpha_2}[1 + h_{21}\xi^{-1} + \ldots], \\ T_3^\infty(\xi) &\sim \xi^{-\frac{1}{4}}[1 + h_{31}\xi^{-\frac{1}{2}} + \ldots]\exp(-m\xi^{\frac{1}{2}}), \\ T_4^\infty(\xi) &\sim \xi^{-\frac{1}{4}}[1 + h_{41}\xi^{-\frac{1}{2}} + \ldots]\exp(m\xi^{\frac{1}{2}}), \end{aligned}\right\} \tag{3.4}$$

where

$$\alpha_1 = -\tfrac{1}{2} + i\beta, \quad \alpha_2 = -\tfrac{1}{2} - i\beta = \bar{\alpha}_1, \quad \beta = \tfrac{1}{2}(\sigma^2 - 1)^{\frac{1}{2}}, \quad m = 2(\gamma\sigma)^{\frac{1}{2}}. \tag{3.5}$$

The first two of these represent waves travelling respectively downward and upward which approximate waves in an inviscid and non-conducting isothermal atmosphere: $T_1^\infty \sim \exp[(\tfrac{1}{2} - i\beta)z]$ and $T_2^\infty \sim \exp[(\tfrac{1}{2} + i\beta)z]$. The third one corresponds to the boundary-layer term, which decays with increasing z like $\exp[-(\gamma\sigma/2\epsilon)^{\frac{1}{2}}z]$. This term would be present if the boundary condition (2.7) were prescribed. It is more convenient to use the lower boundary condition, from which it follows that the solution $T(\xi)$ must behave asymptotically like a linear combination of T_1^∞ and T_2^∞, i.e. $T(\xi) \sim d_1\xi^{\alpha_1} + d_2\xi^{\alpha_2}$. The ratio d_1/d_2 of the coefficients determines the reflexion coefficient.

The point $\xi = 0$ is a regular singular point of (3.3) with characteristic exponents

$$\rho_1 = \tfrac{1}{2} + i\beta^*, \quad \rho_2 = \bar{\rho}_1 = \tfrac{1}{2} - i\beta^*, \quad \rho_3 = \rho_4 = 0, \tag{3.6}$$

where $\beta^* = \frac{1}{2}(\gamma\sigma^2 - 1)^{\frac{1}{2}}$. Consequently, there are four independent solutions of the form

$$\left.\begin{aligned} T_i(\xi) &= \Sigma\, a_n(\rho_i)\,\xi^{n+\rho_i}, \quad a_0 = 1, \quad i = 1, 2, 3,\\ T_4(\xi) &= \Sigma\, a_n'(0)\,\xi^n + T_3(\xi)\log\xi. \end{aligned}\right\} \tag{3.7}$$

Here and subsequently all the sums are infinite sums and a prime indicates differentiation. All the series converge for $|\xi| < \infty$ since $\xi = 0$ and $\xi = \infty$ are the only singular points of (3.3). The first terms of these expansions describe the behaviour of (3.1) for fixed $\epsilon > 0$ and large z:

$$\left.\begin{aligned} T_1(z) &\sim \exp\left[-(\tfrac{1}{2}+i\beta^*)z\right], \quad & T_2(z) &\sim \exp\left[(-\tfrac{1}{2}+i\beta^*)z\right],\\ T_3(z) &\sim \text{constant} + O(e^{-z}), \quad & T_4(z) &\sim z + O(1). \end{aligned}\right\} \tag{3.8}$$

It is clear that only T_4 fails to satisfy the entropy condition (2.9). Thus, the conditions imposed so far are insufficient for determining a unique solution. We shall, therefore, add on the *radiation condition*, which rules out T_1 since the energy associated with it travels downward. Thus, the required solution must be a linear combination of T_2 and T_3:

$$T(\xi) = c_2 T_2(\xi) + c_3 T_3(\xi). \tag{3.9}$$

The coefficients c_2 and c_3 are to be determined so that the asymptotic expansion of $T(\xi)$ does not contain the exponentially increasing term $T_3^\infty(\xi)$.

The behaviour of the velocity oscillations as $z \to \infty$ can be obtained from (2.5*a*) and (3.8), which yield

$$W(z) \sim \text{constant} \times \exp\left(\tfrac{1}{2}+i\beta^*\right)z. \tag{3.10}$$

The velocity oscillations are not inhibited by thermal conduction but the wavelength is affected, and it is evident that energy is carried off to infinity.

We now turn to the problem of finding the asymptotic behaviour (as $\xi \to \infty$) of T_2 and T_3. The coefficients $a_n(\rho_j)$ in the expansions (3.7) are determined from the recursion formula

$$f_0(n+1+\rho_j)\,a_{n+1} + f_1(n+\rho_j)\,a_n = 0, \quad a_0 = 1, \quad j = 1, 2, 3, \tag{3.11}$$

where $$f_0(y) = y^2(y^2 - y + \tfrac{1}{4}\gamma\sigma^2), \quad f_1(y) = -\gamma\sigma(y^2 + y + \tfrac{1}{4}\sigma^2).$$

Making use of the functional relation $\Gamma(y+1) = y\Gamma(y)$ for the gamma function, we obtain the following expression for $a_n(\rho_j)$, $j = 1, 2, 3$:†

$$a_n(\rho_j) = b\,\frac{(\gamma\sigma)^{n+\rho_j}\,\Gamma(n+\rho_j-\alpha_1)\,\Gamma(n+\rho_j-\alpha_2)}{\Gamma^2(n+\rho_j+1)\,\Gamma(n+\rho_j-\beta_1)\,\Gamma(n+\rho_j-\beta_2)}, \tag{3.12}$$

where $$b = b(\rho_j) = \frac{\Gamma^2(\rho_j+1)\,\Gamma(\rho_j-\beta_1)\,\Gamma(\rho_j-\beta_2)}{(\gamma\sigma)^{\rho_j}\,\Gamma(\rho_j-\alpha_1)\,\Gamma(\rho_j-\alpha_2)} \tag{3.13}$$

and $$\beta_1 = -\tfrac{1}{2} + i\beta^*, \quad \beta_2 = -\tfrac{1}{2} - i\beta^* = \overline{\beta_1^*}.$$

The asymptotic behaviour of T_2 and T_3 can now be obtained directly from the results of Ford (1960, chap. VII). Omitting the details of the computations, which can be found in Lyons (1972), we obtain for $j = 1, 2$ and 3

$$T_j(\xi) \sim b_{j1}T_1^\infty + b_{j2}T_2^\infty + b_{j3}T_3^\infty, \quad |\xi| \to \infty, \quad \arg\xi = \tfrac{3}{2}\pi, \tag{3.14}$$

† The expression for $a_n(\rho_1)$ will be needed in §4.

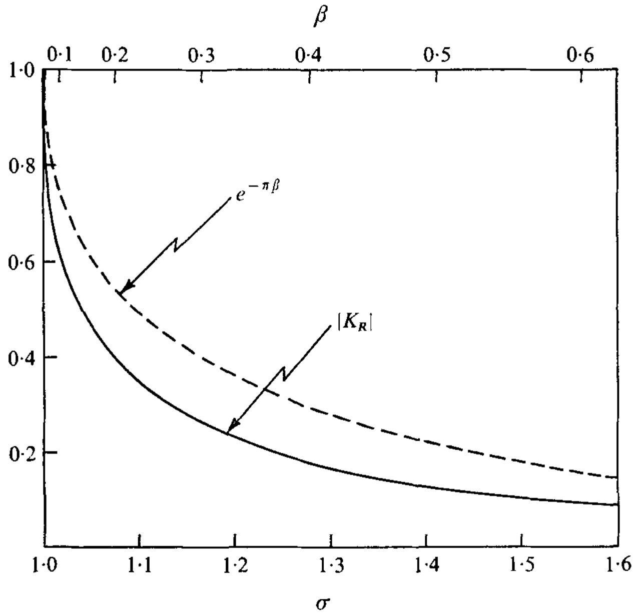

FIGURE 1. Magnitude of the reflexion coefficient for case III as a function of frequency σ, or of the wavenumber β.

where
$$\left.\begin{aligned} b_{j1} &= -b(\rho_j)\,C(\alpha_1,\rho_j), \quad b_{j2} = -b(\rho_j)\,C(\alpha_2,\rho_j), \\ b_{j3} &= ib(\rho_j)\,(\gamma\sigma)^{-\frac{1}{2}}\exp(2\pi i\rho_j)/2\sqrt{\pi}, \\ C(\alpha,\rho) &= \pi(\gamma\sigma)^{\alpha}\exp[-i\pi(\alpha-\rho)]/G(\alpha)\sin\pi(\alpha-\rho), \\ G(\alpha) &= \Gamma^2(\alpha+1)\,\Gamma(\alpha-\beta_1)\,\Gamma(\alpha-\beta_2)/\Gamma(\alpha-\bar{\alpha}). \end{aligned}\right\} \tag{3.15}$$

Thus,
$$T(\xi) \sim (c_2 b_{21}+c_3 b_{31})\,T_1^\infty + (c_2 b_{22}+c_3 b_{32})\,T_2^\infty + (c_2 b_{23}+c_3 b_{33})\,T_3^\infty, \tag{3.16}$$

and, since the coefficient of T_3^∞ must vanish,
$$c_2/c_3 = -b_{33}/b_{23}.$$

This determines T except for a normalizing constant. Returning to the z coordinate, we have in any fixed interval $0 \leqslant z \leqslant z_1$
$$T(z) = \text{constant}\,\{\exp[(\tfrac{1}{2}+i\beta)\,z] + K_R\exp[(\tfrac{1}{2}-i\beta)\,z]\}\,[1+O(\epsilon)], \tag{3.17}$$

where the reflexion coefficient K_R is given by
$$\begin{aligned} K_R &= (i\epsilon)^{-2i\beta}(c_2 b_{21}+c_3 b_{31})/(c_2 b_{22}+c_3 b_{32}) \\ &= (i\epsilon)^{-2i\beta}\frac{C(\alpha_1,\rho_2)\,e^{2\pi i\rho_3} - C(\alpha_1,\rho_3)\,e^{2\pi i\rho_2}}{C(\alpha_2,\rho_2)\,e^{2\pi i\rho_3} - C(\alpha_2,\rho_3)\,e^{2\pi i\rho_2}} \\ &= e^{\pi\beta}\frac{\sinh\pi(\beta^*-\beta)}{\sinh\pi(\beta^*+\beta)}\frac{G(\alpha_2)}{G(\alpha_1)}\exp[2i\beta\log(\gamma\sigma/\epsilon)]. \end{aligned} \tag{3.18}$$

The following picture emerges for small ϵ and $\sigma > 1$. (*a*) In the region where $\epsilon e^z \ll 1$ the solution consists of an incident and a reflected wave, the wavelength being $\lambda = 2\pi/\beta = 4\pi(\sigma^2-1)^{-\frac{1}{2}}$. (*b*) In the region where $\epsilon e^z \gg 1$ the solution behaves like an outgoing wave with wavelength $\lambda^* = 2\pi/\beta^* = 4\pi(\gamma\sigma^2-1)^{-\frac{1}{2}} < \lambda$. (*c*) The reflexion takes place in the region where $\epsilon e^z \sim 1$. (*d*) As $\epsilon \to 0$ the solution $T(z)$ approaches a limit in a co-ordinate system which shifts so that ϵe^z = constant.

It should be noted that

$$|K_R| = e^{\pi\beta}\frac{\sinh\pi(\beta^*-\beta)}{\sinh\pi(\beta^*+\beta)}$$
$$= e^{-\pi\beta}\frac{1-\exp[-2\pi(\beta^*-\beta)]}{1-\exp[-2\pi(\beta^*+\beta)]} < e^{-\pi\beta}. \tag{3.19}$$

Thus $|K_R|$ is smaller than it is for case II. As in case II, $|K_R| \to 1$ as $\beta \to 0$ (total reflexion) and $|K_R| \to 0$ as $\beta \to \infty$. Thus, the radiation condition becomes more accurate as the wavelength decreases. A plot of $|K_R|$ as a function of σ is shown in figure 1.

The exponential growth of the velocity casts doubt on the validity of the results. Hence, we turn to the problem in which both viscosity and thermal conduction are taken into account.

4. Oscillations of a viscous and thermally conducting atmosphere

We shall now consider the boundary-value problem for the differential equation (2.6) subject to the lower boundary condition and conditions (2.8) and (2.9). Introducing the independent variable ξ by means of (3.2) transforms (2.6) into

$$\{\sigma Pr\,\theta^2(\theta-1)(\theta-2)-\sigma^2 Pr\,\xi(\theta^2-\theta)-3\xi\theta^2(\theta^2-\theta+\tfrac{1}{4}\gamma\sigma^2)$$
$$+3\gamma\sigma\xi^2(\theta^2+\theta+\tfrac{1}{4}\sigma^2)\}T = 0. \tag{4.1}$$

The point $\xi = 0$ is a regular singular point of (4.1) with characteristic exponents $\rho_1 = 2$, $\rho_2 = 1$ and $\rho_3 = \rho_4 = 0$, and there are four solutions, which, in a neighbourhood of $\xi = 0$, can be represented in the form

$$\left.\begin{aligned} T_1(\xi) &= \Sigma\, a_n(2)\,\xi^{n+2}, \quad T_2(\xi) = \Sigma\, a_n'(1)\,\xi^{n+1}+T_1(\xi)\log\xi,\\ T_3(\xi) &= \Sigma\, a_n''(0)\,\xi^n, \quad T_4(\xi) = \Sigma a_n'''(0)\,\xi^n+3T_3(\xi)\log\xi, \end{aligned}\right\} \tag{4.2}$$

where the coefficients $a_n(\rho_i)$ are determined from a three-term recursion formula and $a_0(\rho) = \rho^2(\rho-1)$. Returning to the z co-ordinate, we obtain for large z

$$T_1(z) = O(e^{-2z}), \quad T_2(z) = O(e^{-z}), \quad T_3(z) = O(1), \quad T_4(z) = O(z). \tag{4.3}$$

The velocity amplitudes corresponding to these can be obtained from (2.5*a*), which results in

$$W_1'(z) = O(e^{-z}), \quad W_2'(z) = O(1), \quad W_3'(z) = O(e^{-z}), \quad W_4'(z) = O(z). \tag{4.4}$$

Since W_2 and W_4 do not satisfy (2.8) and T_4 does not satisfy (2.9), the solution must have the form

$$T(z) = c_1T_1(z)+c_3T_3(z), \quad W(z) = c_1'W_1(z)+c_3'W_3(z). \tag{4.5}$$

The point $\xi = \infty$ is an irregular singular point of (4.1), and to first order the asymptotic behaviour of the solutions is governed by the same terms as in the differential equation (3.3) for case III. Thus, there are four solutions with asymptotic expansions of the form (3.4). The lower boundary condition requires the coefficient of T_3^∞ in the asymptotic expansion of the solution $T(\xi)$ to vanish, and this suffices, in general, to determine the ratio of the coefficients c_1 and c_3 in (4.5). Thus, the boundary-value problem has a unique solution except for a normalizing constant. It should be noted that, in contrast to case III, both T and W approach constant values as $z \to \infty$, so that the linearizing assumption is not violated as it is in cases I and III. Since the density decays exponentially with z, the energy density also decays exponentially.

It is difficult to obtain the asymptotic behaviour of $T_1(\xi)$ and $T_3(\xi)$ from the expansions (4.2) since the coefficients a_n satisfy a three-term recursion formula. We shall limit ourselves, therefore, to the case of small Prandtl number, which leads to a singular perturbation problem.

Let $R(\xi)$ denote the 'inner' approximation, which is obtained by setting $Pr = 0$ in (4.1). The resulting differential equation,

$$\{\gamma\sigma\xi(\theta^2+\theta+\tfrac{1}{4}\sigma^2)-\theta^2(\theta^2-\theta+\tfrac{1}{4}\gamma\sigma^2)\}R = 0, \tag{4.6}$$

is precisely the one for case III. As we have seen, the solutions of (4.6) have the same behaviour to first order as those of (4.1) for large ξ, but the same is not true near $\xi = 0$, as is evident from (3.7) and (4.2). However, $R(\xi)$ will approximate the required solution of (4.1) uniformly as $Pr \to 0$ on $A \leqslant |\xi| < \infty$, $\arg\xi = \frac{3}{2}\pi$, for any positive A.

To obtain the 'outer' approximation we introduce the stretching $\xi = Pr\,\eta$, which transforms the singular point at $\xi = \frac{1}{3}\sigma Pr$ into a stationary one at $\eta = \frac{1}{3}\sigma$ and the differential equation (4.1) into

$$\{\sigma(\theta-1)(\theta-2)-3\eta(\theta^2-\theta+\tfrac{1}{4}\gamma\sigma^2)\}\theta^2T+Pr\,\eta\{3\gamma\sigma\eta(\theta^2+\theta+\tfrac{1}{4}\sigma^2) - \sigma^2(\theta^2-\theta)\}T = 0. \tag{4.7}$$

The velocity amplitude W can be obtained from (2.5a), which becomes

$$-\frac{i}{\eta\,Pr}\theta^2T+i\sigma T+\frac{\gamma-1}{\gamma}\theta W = 0. \tag{4.8}$$

Letting $T(\eta) = Pr\,S(\eta)$ and setting $Pr = 0$ yields

$$\{\sigma(\theta-1)(\theta-2)-3\eta(\theta^2-\theta+\tfrac{1}{4}\gamma\sigma^2)\}\theta^2S = 0, \tag{4.9}$$

$$\theta W = [i\gamma/(\gamma-1)]\,\eta^{-1}\theta^2S. \tag{4.10}$$

The solutions of (4.9) will approximate those of (4.7) if $\xi = Pr\,\eta$ is small. Since $\eta \to \infty$ for any fixed positive $|\xi|$ as $Pr \to 0$, the asymptotic behaviour of $S(\eta)$ as $\eta \to \infty$ must be matched with the behaviour of $R(\xi)$ as $\xi \to 0$.

The outer approximation $S(\eta)$ must satisfy (4.9) and the conditions (2.8) and (2.9). The point $\eta = 0$ is a regular singular point of (4.9), with characteristic exponents $\rho_1 = 2$, $\rho_2 = 1$ and $\rho_3 = \rho_4 = 0$. Corresponding to the first two there are two solutions of the form

$$S_1(\eta) = \Sigma\,a_n(2)\,\eta^{n+2}, \quad S_2(\eta) = \Sigma\,a_n'(1)\,\eta^{n+1}+S_1(\eta)\log\eta, \tag{4.11a}$$

where the series converge for $|\eta| < \frac{1}{3}\sigma$. The solutions corresponding to ρ_3 and ρ_4 are simply

$$S_3(\eta) = 1, \quad S_4(\eta) = \log\eta. \tag{4.11b}$$

It is easy to see from (4.10) that only S_1 and S_3 satisfy all the conditions, so that

$$S(\eta) = c_1 S_1(\eta) + c_3 S_3(\eta) = c_1 S_1(\eta) + c_3. \tag{4.12}$$

The point $\eta = \infty$ is also a regular singular point of (4.9), and there are four solutions of the form

$$\left.\begin{aligned} S_j^\infty(\eta) &= \eta^{1+\beta_j} \Sigma h_{jk} \eta^{-k}, \quad j = 1, 2, \\ S_3^\infty(\eta) &= S_3(\eta) = 1, \quad S_4^\infty(\eta) = S_4(\eta) = \log \eta. \end{aligned}\right\} \tag{4.13}$$

The asymptotic behaviour of $S_1(\eta)$ as $\eta \to \infty$ can now be deduced from the first theorem of Ford (1960, chap. I). Omitting the details of the computations, which can be found in Lyons (1972), one obtains

$$S_1(\eta) = d_1 S_1^\infty(\eta) + d_2 S_2^\infty(\eta) + d_3 S_3^\infty(\eta), \quad |\eta| > \tfrac{1}{3}\sigma, \tag{4.14}$$

where

$$\left.\begin{aligned} d_1 &= b(\rho_1)\, i\pi D(\beta_1) \exp(\pi\beta^*)/\cosh \pi\beta^*, \\ d_2 &= b(\rho_1)\, i\pi D(\beta_2) \exp(-\pi\beta^*)/\cosh \pi\beta^*, \\ d_3 &= b(\rho_1)/b(0), \\ D(\beta) &= (3/\sigma)^{1+\beta} \beta \Gamma(\beta - \bar{\beta})/\Gamma^2(\beta + 2), \\ b(\rho) &= \Gamma^2(\rho+1)/[(3/\sigma)^\rho \Gamma(\rho - 1 - \beta_1)\Gamma(\rho - 1 - \beta_2)]. \end{aligned}\right\} \tag{4.15}$$

Consequently,

$$S(\eta) = c_1 d_1 S_1^\infty(\eta) + c_1 d_2 S_2^\infty(\eta) + (c_1 d_3 + c_3), \quad |\eta| > \tfrac{1}{3}\sigma. \tag{4.16}$$

Retaining the most significant terms and returning to the ξ variable yields the result

$$S(\xi) \sim c_1 d_1 (\xi/Pr)^{\frac{1}{2}+i\beta^*} + c_1 d_2 (\xi/Pr)^{\frac{1}{2}-i\beta^*} + c_1 d_3 + c_3, \tag{4.17}$$

where the ratio c_3/c_1 is to be determined from the matching procedure.

We now turn to the inner approximation $R(\xi)$. The problem for the differential equation (4.6) has already been considered in §3. In view of (4.17),

$$R(\xi) = \tilde{a}_1 T_1(\xi) + \tilde{a}_2 T_2(\xi) + \tilde{a}_3 T_3(\xi), \tag{4.18}$$

where the T_j are given in (3.7). The procedure for finding the asymptotic behaviour is the same as for case III, and the details need not be repeated. The result is different, of course, because T_1 must be included in (4.18) since the radiation condition is not applicable any more. The asymptotic behaviour of $R(\xi)$ is given by

$$\begin{aligned} R(\xi) \sim (\tilde{a}_1 b_{11} + \tilde{a}_2 b_{21} + \tilde{a}_3 b_{31}) T_1^\infty + (\tilde{a}_1 b_{12} + \tilde{a}_2 b_{22} + \tilde{a}_3 b_{32}) T_2^\infty \\ + (\tilde{a}_1 b_{13} + \tilde{a}_2 b_{23} + \tilde{a}_3 b_{33}) T_3^\infty, \end{aligned} \tag{4.19}$$

where the coefficients b_{ij} are specified in (3.15). The vanishing of the coefficient of T_3^∞ follows from the lower boundary condition, and this yields a relation between the $\tilde{a}_j$:

$$\exp(-2\pi\beta^*)\tilde{a}_1 + \exp(2\pi\beta^*)\tilde{a}_2 - \tilde{a}_3 = 0. \tag{4.20}$$

Retaining only the most significant terms, we obtain

$$R(\xi) \sim \tilde{a}_1 \xi^{\frac{1}{2}+i\beta^*} + \tilde{a}_2 \xi^{\frac{1}{2}-i\beta^*} + \tilde{a}_3 \tag{4.21}$$

for small ξ, and comparison with (4.17) results in

$$\tilde{a}_1 = c_1 d_1 Pr^{-\frac{1}{2}-i\beta^*}, \quad \tilde{a}_2 = c_1 d_2 Pr^{-\frac{1}{2}+i\beta^*}, \quad \tilde{a}_3 = c_1 d_3 + c_3. \tag{4.22a, b, c}$$

The four relations (4.20) and (4.22) suffice to determine all the coefficients to within a normalizing constant. For example, if $\tilde{a}_1$ is prescribed, one can compute c_1 and $\tilde{a}_2$ from (4.22*a*, *b*), then $\tilde{a}_3$ from (4.20), and finally c_3 from (4.22*c*).

The behaviour of the solution can now be deduced from (4.19), (4.21) and (4.12). The first of these is valid in the region where $\epsilon e^{+z} \ll 1$, and the effects of viscosity and thermal conduction are negligible. The solution consists of an upward- and downward-travelling (reflected) wave of wavelength $2\pi/\beta$. The ratio of their amplitudes is the reflexion coefficient K_R, which is given by

$$K_R = \exp(-\pi\beta - 2i\beta\log\epsilon)\frac{C(\alpha_2)}{C(\alpha_1)}\frac{V_1 - V_2 e^{2i\phi} Pr^{-2i\beta^*}}{V_3 - V_4 e^{2i\phi} Pr^{-2i\beta^*}}, \tag{4.23}$$

where
$$\left.\begin{aligned} V_1 &= e^{2\pi\beta^*} - e^{2\pi\beta}, \quad & V_2 &= e^{2\pi\beta} - e^{-2\pi\beta^*}, \\ V_3 &= e^{2\pi\beta^*} - e^{-2\pi\beta}, \quad & V_4 &= e^{-2\pi\beta} - e^{-2\pi\beta^*}, \end{aligned}\right\} \tag{4.24}$$

$$\phi = -\beta^*\log(\tfrac{1}{3}\gamma\sigma^2) + \arg\{\beta_1\Gamma(\beta_1 - \beta_2)\Gamma(\beta_2 + 1 - \alpha_1)\Gamma(\beta_2 + 1 - \alpha_2)/\Gamma(\beta_2 + 1 - \beta_1)\}$$

and
$$C(\alpha) = (\gamma\sigma)^{-\alpha}\Gamma^2(\alpha + 1)\Gamma(\alpha - \beta_1)\Gamma(\alpha - \beta_2)/\Gamma(\alpha - \overline{\alpha}).$$

Consequently,
$$|K_R| = e^{-\pi\beta}\left|\frac{V_1 - V_2 e^{2i\phi} Pr^{-2i\beta^*}}{V_3 - V_4 e^{2i\phi} Pr^{-2i\beta^*}}\right|. \tag{4.25}$$

It is useful to note that the maximum value of $|K_R| = e^{-\pi\beta}(V_1 + V_2)/(V_3 + V_4) = e^{-\pi\beta}$.

The reflexion process is now more complicated than for cases II and III because of the existence of two reflecting layers. The lower one, in the vicinity of $z = -\log\epsilon$, is due to the effect of thermal conduction, while the upper one, near $z = -\log(\epsilon Pr)$, is caused by the viscosity. The coefficient K_R describes the combined effect of both layers on the wave motion below the lower one. Between the two layers the wavelength changes to $2\pi/\beta^*$, and the upper layer is displaced upward as $Pr \to 0$. Thus, increasing $\log Pr$ by π/β^* shifts the upper layer by half a wavelength, which should leave K_R unchanged. The periodicity of K_R is indeed evident from (4.23). In the region above the upper layer ($\epsilon Pr\, e^z \gg 1$) the wave motion is damped.

The periodicity of $|K_R|$ with $\log Pr$ is in marked contrast to the result of case III, for which $|K_R| = (V_1/V_3)\, e^{-\pi\beta}$. The conclusions of case III should, therefore, be considered erroneous. However, the mean-square value of the difference between $(V_1/V_3)\, e^{-\pi\beta}$ and $|K_R|$ turns out to be small compared with $e^{-\pi\beta}$ and tends to zero as $\beta \to 0$, which suggests that the results of case III are not completely meaningless. The situation is illustrated by the graphs in figures 2(*a*) and (*b*).

5. Numerical results

The results of the previous section are asymptotically valid as $Pr \to 0$. In order to determine the range of Pr in which they are reasonably accurate, the boundary-value problem was solved numerically for several different values of Pr and β. For the numerical integration it is convenient to deal with the system (2.5) with boundary conditions (2.7) at $z = 0$ and the condition

$$DW, DT \to 0 \quad \text{as} \quad z \to \infty. \tag{5.1}$$

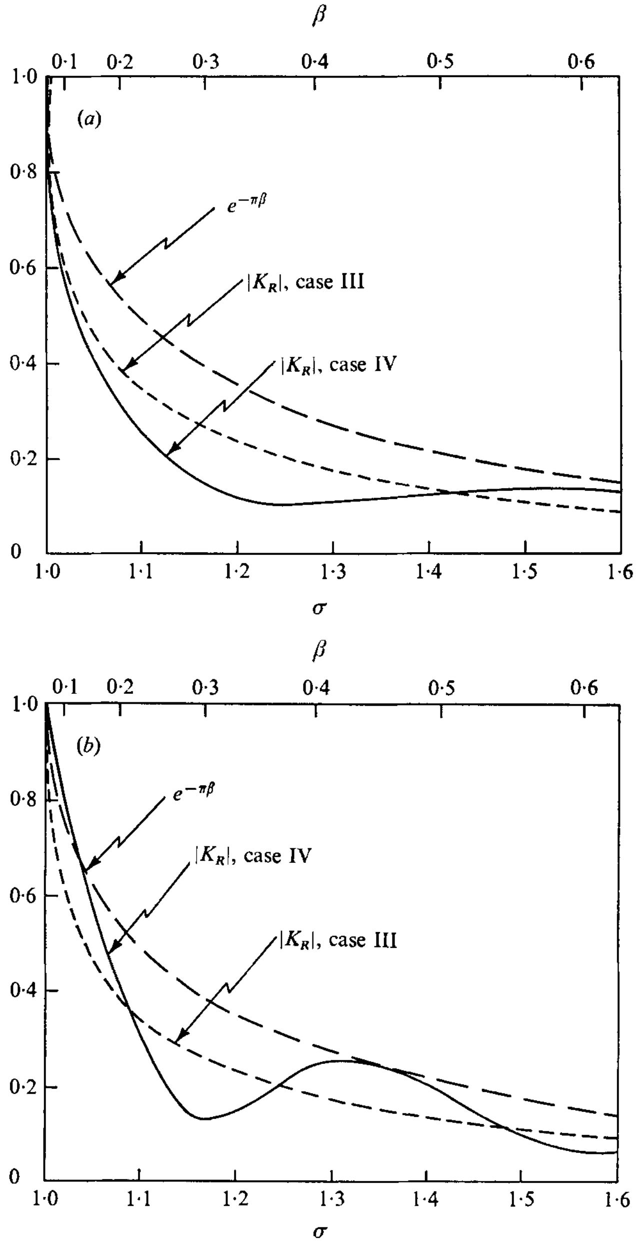

FIGURE 2. Magnitude of the reflexion coefficient for cases III and IV for (*a*) $Pr = 10^{-2}$ and (*b*) $Pr = 10^{-5}$.

The last condition is used in place of (2.8) and (2.9) since we have shown that T and W both approach constant values as $z \to \infty$.

Using centred differences, we can replace the system of differential equations (2.5) by a set of difference equations

$$\mathbf{A}_n \mathbf{\Phi}_{n+1} + \mathbf{B}_n \mathbf{\Phi}_n + \mathbf{C}_n \mathbf{\Phi}_{n-1} = 0, \tag{5.2}$$

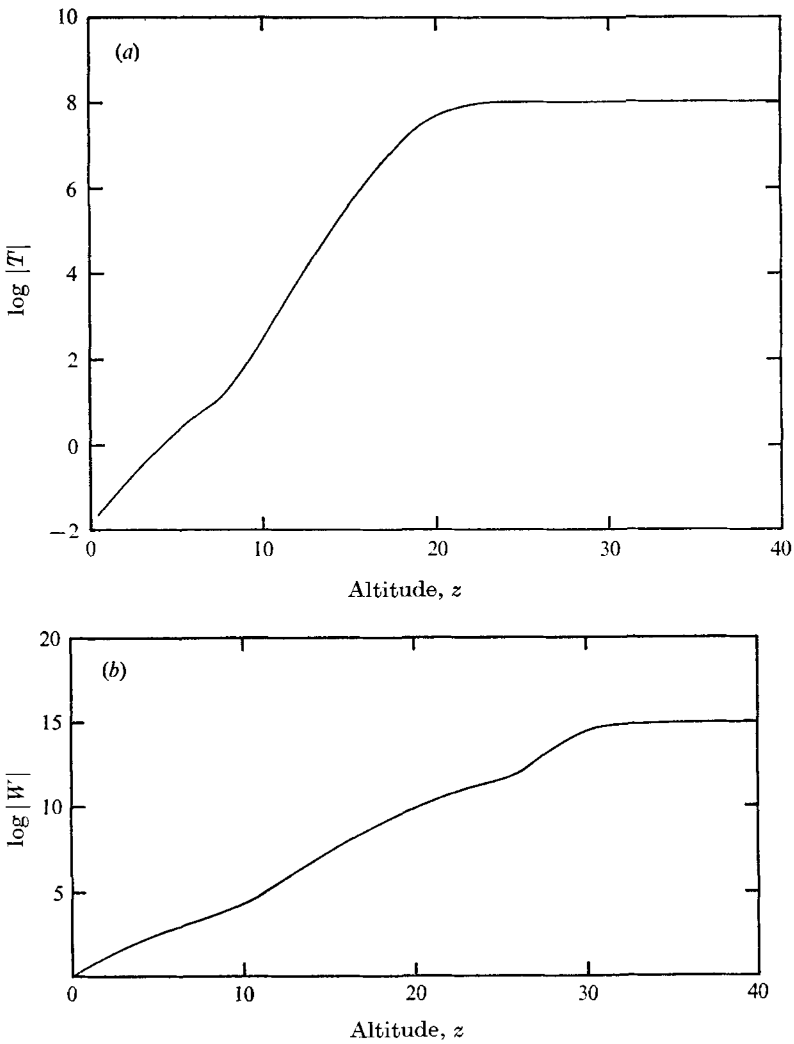

FIGURE 3. Logarithm of (a) the temperature oscillation amplitude and (b) the velocity oscillation amplitude; $Pr = 10^{-5}$, $\sigma = 1{\cdot}084$.

where $\mathbf{\Phi}_n$ is the column vector with components $iT(z_n)$ and $W(z_n)$, and $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$ are 2×2 matrices. The system was solved in an interval $0 \leqslant z \leqslant L$ sufficiently large to allow T and W to reach their limiting values, and the boundary condition (5.1) was replaced by $\mathbf{\Phi}_N = \mathbf{\Phi}_{N-1}$, where N is the index of the point $z = L$. The problem was solved by the standard method in which the $\mathbf{\Phi}_n$ are computed from a linear equation $\mathbf{\Phi}_n = \boldsymbol{\alpha}_n \mathbf{\Phi}_{n+1} + \boldsymbol{\beta}_n$ by backward integration, while the matrices

$$\boldsymbol{\alpha}_n = -[\mathbf{B}_n + \mathbf{C}_n \boldsymbol{\alpha}_{n-1}]^{-1} \mathbf{A}_n,$$

and the vectors

$$\boldsymbol{\beta}_n = -[\mathbf{B}_n + \mathbf{C}_n \boldsymbol{\alpha}_{n-1}]^{-1} \mathbf{C}_n \boldsymbol{\beta}_{n-1}$$

are computed by forward integration (see, for example, Richtmyer & Morton 1957, pp. 198–201; or Lindzen 1970, pp. 326–330).

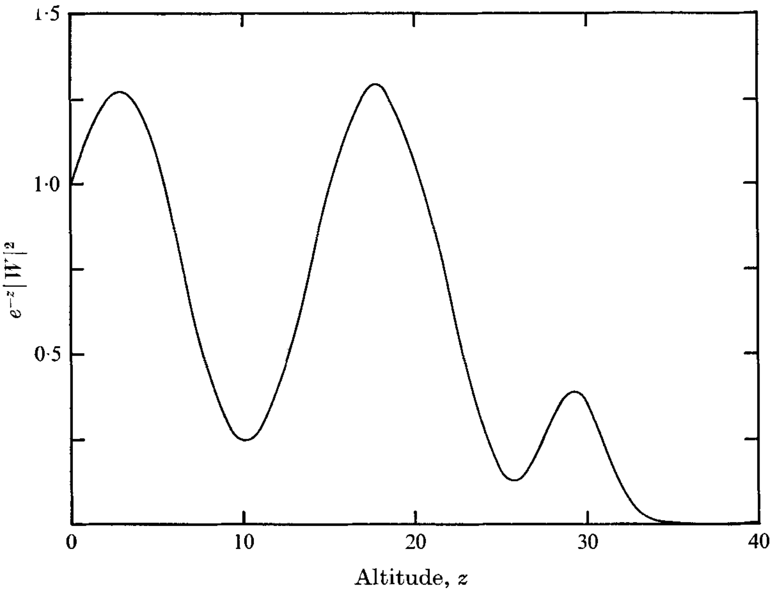

FIGURE 4. Average kinetic energy; $Pr = 10^{-5}$, $\sigma = 1{\cdot}084$.

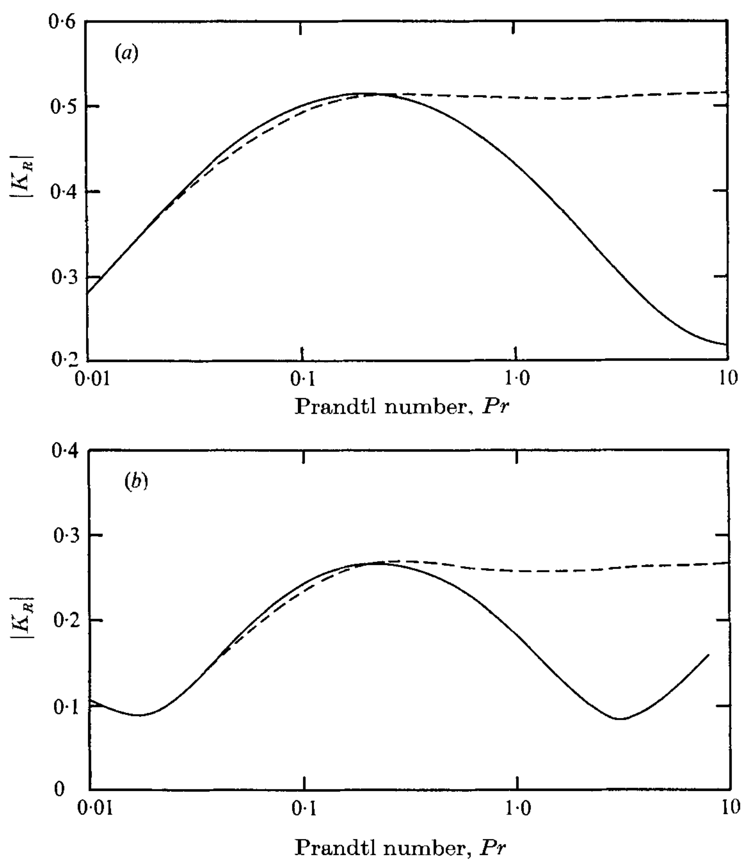

FIGURE 5. Magnitude of the reflexion coefficient as a function of the Prandtl number. ——, asymptotic results; ----, numerical results. (*a*) $\sigma = 1{\cdot}084$. (*b*) $\sigma = 1{\cdot}305$.

The problem was solved with $\epsilon = 10^{-8}$, a sufficiently small value to test the asymptotic formulae, for two different values of the wavelength $\lambda = 2\pi/\beta = 30$ and 15 (at which the reflexion is substantial) and for various values of Pr. A value of 40 or 50 was more than adequate for L, and a non-uniform mesh size was needed because of the boundary layer near $z = 0$. For $z > 0{\cdot}2$ the mesh size $\Delta z = 0{\cdot}1$ was used, while in the boundary layer 1000 points were found to be sufficient with $\Delta z = 10^{-6}$. Between the two regions the mesh size was gradually increased to provide a smooth transition.

Sample solutions are shown in figures 3(*a*) and (*b*). The plot of the kinetic energy density (figure 4) clearly indicates the transition from one wavelength to another one, and allows one to calculate the magnitude $|K_{RN}|$ of the reflexion coefficient. Letting M and m denote the maximum and minimum values of the oscillation amplitude, and $r = (M/m)^{\frac{1}{2}}$, $|K_{RN}|$ can be computed from

$$|K_{RN}| = (r-1)/(r+1). \tag{5.3}$$

The values computed from the graphs were found to agree with the values obtained from (4.25) to about three places.

A comparison of the values of the magnitude of the reflexion coefficient obtained from the computations and from the asymptotic formula is shown in figures 5(*a*) and (*b*). They are in good agreement for $Pr < 0{\cdot}02$ and in fair agreement even up to $Pr = 0{\cdot}2$. Above that value $|K_{RN}|$ is essentially constant and close to the value $e^{-\pi\beta}$, the asymptotic result for case II. A reasonable approximation can be obtained, therefore, merely by piecing together the asymptotic results for cases II and IV.

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